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Multiscale Formalism for Correlation Functions of Fermions. Infrared Analysis of the Tridimensional Gross–Neveu Model

Emmanuel Pereira,[1] Aldo Procacci,[2] and Michael O'Carroll[1]

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We present a multiscale formalism for fermionic systems (with a smooth UV cutoff) establishing a trivial link between the correlation functions and the effective potential flow, and study the k-point truncated functions of the tridimensional Gross–Neveu model. A new efficient method is used to bound these correlation functions and show polynomial tree decay for long distances. We are guided by a block lattice mechanism with a property of orthogonality between terms in different scales, which leads to simple formulas for the correlations.

KEY WORDS: Correlation functions; fermions; infrared analysis; tridimensional Gross–Neveu model.

1. INTRODUCTION

Problems with many scales of length have been rigorously studied via renormalization group (RG) methods for a long time: bosonic theories such as Φ_4^4 [GK1], the dipole gas [BY, GK2], classical Heisenberg model [B], fermionic systems such as Gross–Neveu [FMRS, GK3] and Fermi-liquids [BG1, BGPS, FMRT, FST] are well-known examples. According to the approach and the model, several techniques have been developed and used within the RG mechanism, such as polymer and tree expansions, small and large field analysis, etc., involving always intricate propositions, sometimes hard to prove.

[1] Departamento de Física-ICEx, UFMG, CP 702, Belo Horizonte MG 30.161-970, Brazil; e-mail: emmanuel@fisica.ufmg.br, ocarroll@fisica.ufmg.br.

[2] Departamento de Matemática-ICEx, UFMG, CP 702, Belo Horizonte MG 30.161-970, Brazil; e-mail: aldo@mat.ufmg.br.

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In particular, the correlation functions of such systems have been also treated via the RG method [BGPS, BO, DH, FMRS, GK4, IM, MS], considered the "next problem" after controlling the effective action flow, and usually requiring a more considerable effort, in spite of the elaborate analysis already presented in the action flow study.

Recently, having in mind the search for simplifications in the treatment of correlation functions, we studied the well-known lattice dipole gas [OP, PO]. We noted the association of the block RG formalism carefully developed in [GK1] (Wilson–Kadanoff RG) with a free propagator decomposition with a property of orthogonality between terms on different scales, and this property allowed us to establish a trivial link between the potential flow and the correlation functions. In short, the Wilson–Kadanoff RG appeared as an effective tool for the study of correlation functions of these bosonic models. In [P, PP2] we derive (using, say, a non standard block RG) a lattice mechanism for fermionic systems with similar properties. Here, and throughout this paper, for fermionic systems we mean fermions with the interaction given by a quadratic term with a derivative plus perturbations, such as in a wide class of models: Gross–Neveu, Fermi-liquids, etc.

In the present article, using the lattice block formalism as a guide, we extend our mechanism to obtain a representation for the correlation functions of fermions using a standard RG (presented, e.g., in [BG1, 2], [BGPS], and references there in), without the orthogonality property but avoiding some technical problems of the block transformation. We also establish and control (infrared analysis) *all the k-point* truncated correlation functions of the tridimensional Gross–Neveu model.

In our representation the truncated k-point function has the general structure of summed perturbation theory, i.e., a blob with k propagators attached. In the tridimensional Gross–Neveu model considered here, in the limit of an infinite number of RG transformations, the blob is given by k-field derivatives of the effective potential at zero field and the propagators behave like free ones for large distances, up to the factor of a wavefunction renormalization.

We give a new method for bounding the correlation functions and establishing decay. We use a previously obtained multi-scale representation of the effective potential (see [PP1]) with a pointwise bound of the kernel of the kth field derivative (the blob in the correlation function). The kernels of the k-derivative are bounded by a sum of tree terms each of which is pointwise bounded. We then bound a tree term of the correlation function in a manner similar to the bounding of a term in the Born expansion of non-relativistic quantum mechanical potential scattering.

To understand why we need a multi-scale scheme to solve the problem proposed in the present paper, we make some comments. Roughly speaking, the tridimensional Gross–Neveu can be viewed as a sort of fermionic version of the dipole gas in $d \geqslant 2$ dimensions. The latter is a very studied problem of statistical mechanics (including rigorous RG analysis [BY, DH, GK2, PO]), consisting in a gas of classical particles interacting through a two-body stable but not absolutely integrable potential. The rigorous RG analysis of the dipole gas is performed, in general, by mapping the model (through a Sine–Gordon transformation) into a bosonic field theory. The action of this bosonic model is given by a kinetic marginal term plus a small irrelevant perturbation: the relevant mass term cannot be generated in the RG flow due to the symmetry of the initial action (its dependence on derivative fields). Hence, the parallel with our model: the action of our fermionic Gross–Neveu model has also the structure of a kinetic marginal term plus an irrelevant (quartic) perturbation, and the relevant mass term cannot be generated during the RG flow because of the symmetry properties of the initial action. Thus, as in the case of the dipole gas, to obtain some rigorous results such as the absolute convergence of the perturbative expansion in λ (uniform in the volume Λ) for the pressure, effective potential kernels, etc. a treatment involving just one step integration (all scales at once) does not work, unless one is able to exploit suitable cancellations without introducing dangerous combinatorial factors. Due to the difficulty of the latter task, a direct proof of the analicity of the pressure for the dipole gas is still missing. In relation to our fermionic model, the machinery of the scale per scale RG analysis (and the consequent resummation) provides the standard (and, as far as we know, unique) tool to handle these kind of cancellations.

We emphasize that, although analyzing a simple model (with canonical decay, and just one relevant parameter) we present here a general approach for the study of correlation functions. In fact, we expect to use a similar formalism in the study of more complicated models, even those with non-canonical scaling: the one-dimensional Fermi-liquid (with non-canonical scaling), for instance, has been controlled in [BGPS] by a RG formalism where the renormalization of the propagator, at each RG step, is carried out after manipulations similar to those considered in the study of our effective potential [PP1]. However, in [BGPS], only the two-point function has been treated.

The article is organized as follows. In Section 2 we derive the formalism for the correlations using a more standard RG (with a smooth regularization). Section 3 is devoted to the infrared analysis of the k-point truncated functions for the tridimensional Gross–Neveu model.

2. THE FORMALISM FOR GENERATING AND CORRELATION FUNCTIONS

The fermionic multiscale mechanism to be presented here is inspired on the block spin lattice formalism for fermions [PP2, P], which, say, mimics the bosonic representation [OP, PO], where suitable structures naturally appear with the RG applications. In these previous works, we show that properly using specific block RG transformations, we can write the generating function (for some models) in terms of a "local" effective action (which goes, say, to a Gaussian fixed point), a "small" irrelevant perturbative potential, and two new propagators (denoted by $\tilde{P}_n$ and $\tilde{G}_n$) written as a sum of massive interactions, living in different momentum scales.

The usefulness and simplicity of the representation described there [PP2, P, OP, PO] must be emphasized: for the two point function the dominant term is isolated in $\tilde{P}_n$, and the formulas for the $2k$ point truncated functions are given by field derivatives of the "irrelevant" effective potential at zero field in a combination with $\tilde{G}_n$. Due to the orthogonality property there is no mix between scales, which makes simple the expressions of $\tilde{P}_n$ and $\tilde{G}_n$, with long distance behavior depending only on the flow of the running coupling constants, thus establishing a trivial link between the effective potential and the correlation function theory for these lattice systems.

However, the treatment of fermions on a lattice involves some obstacles: the block RG, for instance, due to the average over blocks, carries technical difficulties such as the lack of translational invariance of several operators into the structure of the effective potential, leading to an extra work in the RG flow analysis. For those reasons, fermions have been mostly treated with smooth cuttoffs, avoiding the lattice and the block RG [BG1, BGPS, FMRT, GK3]. Thus, the search of a good representation for the correlation functions within this scenario (continuum approach and smooth RG) becomes an interesting problem.

That is our aim now: we will extend the block spin formalism [PP2] to the continuum case. The basic structure will be preserved but the orthogonality property will be lost, leading to extra analytical work to bound the operators corresponding to $\tilde{P}_n$ and $\tilde{G}_n$ (fortunately, not so hard due to the asymptotic freedom of the tridimensional Gross–Neveu model in the IR sector).

For the action (from now on, restricting our attention to the tridimensional Gross–Neveu model) we take $\mathscr{H} = \mathscr{H}_0 + V_0$, with

$$\mathscr{H}_0 = \int_\Lambda dx\, \bar{\psi}(x)(i\bar{\partial}_{(\geqslant 0)}\psi)(x), \qquad V_0 = \lambda \int_\Lambda [\bar{\psi}(x)\, \psi(x)]^2\, dx \tag{2.1}$$

where $x \in \Lambda \subset \mathbb{R}^{2+1}$, Λ a periodic box, $\partial\!\!\!/ = \sum_\mu \partial_\mu \gamma^\mu$, γ^μ are 4×4 anti-hermitian matrices satisfying $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\delta^{\mu\nu}$. The Grassmann fields $\bar{\psi}(x)$, $\psi(x)$ (with supressed spinor indices $\alpha = 1, 2, 3, 4$; number of colours $N = 1$) are defined as $\psi(x) = \sum_p \tilde{\psi}(p)\, e^{ipx}$, with p assuming discrete values (Λ is a periodic box), and so, the system has a countable number of Grassmann variables (finite number if one still considers $|p|$ bounded). Anyway, we keep the notation considering integrals over p. The free propagator, with a smooth UV cutoff, is given by

$$[i\partial\!\!\!/_{(\geqslant 0)}]^{-1}\,(x-y) = g_{(\geqslant 0)}(x-y) = \frac{1}{(2\pi)^3} \int d^3p\; e^{ip(x-y)}\, \frac{p\!\!\!/}{p^2}\, e^{-p^2} \tag{2.2}$$

or in terms of an IR and UV regularized covariance

$$\begin{aligned} g_{(\geqslant 0)}(x-y) &= \lim_{M \to \infty} g_{(M,0)}(x-y) \\ &= \lim_{M \to \infty} \frac{1}{(2\pi)^3} \int d^3p\; e^{ip(x-y)}\, \frac{p\!\!\!/}{p^2}\, [e^{-p^2} - e^{-L^{2M}p^2}] \end{aligned} \tag{2.3}$$

The notation $(\geqslant 0)$, which shall be clear ahead, means that there is an UV cutoff at scale L^0. We have

$$|g_{(\geqslant 0)}(x-y)| \leqslant \frac{c}{1 + |x-y|^2} \tag{2.4}$$

(where the bound is, here and throughout the paper, for each term of the 4×4 matrix related to the spinor indices α).

The multiscale formalism starts with the standard multiscale decomposition

$$g_{(\geqslant 0)}(x-y) = \sum_{j=0}^{\infty} g_j(x-y) \tag{2.5}$$

where

$$g_j(x-y) = \int \frac{d^3p}{(2\pi)^3}\, \frac{(e^{-L^{2j}p^2} - e^{-L^{2j+2}p^2})}{p^2}\, p\!\!\!/ e^{ip(x-y)} \tag{2.6}$$

a (translational invariant) massive multiscale decomposition, since

$$g_j(x) = L^{-2j} C(L^{-j}x), \qquad C(x) = \int \frac{d^3p}{(2\pi)^3}\, \frac{(e^{-p^2} - e^{-L^2p^2})}{p^2}\, p\!\!\!/ e^{ip\cdot x}$$

$$|C(x)| \leqslant \text{const.} \exp[-\beta\, |x|] \tag{2.7}$$

($L > 1$, β a positive constant) and so

$$|g_j(x)| \leqslant \text{const. } L^{-2j} \exp[-\beta L^{-j} |x|] \tag{2.8}$$

This expression is similar to (2.10) in [PP2], with g_j (now, translational invariant) corresponding to $\tilde{\Gamma}_j = M_j \Gamma_j M_j^\dagger$ (not translational invariant). With this decomposition the normalized Gaussian measure

$$P^{(\geqslant 0)}(d\psi) = \frac{\prod_{x \in \Lambda} d\bar{\psi}_x \, d\psi_x \, e^{-\int_\Lambda dx \, \bar{\psi}_x (i\not{\partial}_{(\geqslant 0)}) \psi_x}}{\int \prod_{x \in \Lambda} d\bar{\psi}_x \, d\psi_x \, e^{-\int_\Lambda dx \, \bar{\psi}_x (i\not{\partial}_{(\geqslant 0)}) \psi_x}} \tag{2.9}$$

(with covariance $g_{(\geqslant 0)}(x-y)$) may be written as

$$P^{(\geqslant 0)}(d\psi) = \prod_{j=0}^{\infty} P(d\psi_j), \qquad \psi_{(\geqslant 0)} = \sum_{j=0}^{\infty} \psi_j \tag{2.10}$$

or, in a regularized version, $P^{(M,0)}(d\psi) = \prod_{j=0}^{M} P(d\psi_j)$, $\psi_{(M,0)} = \sum_{j=0}^{M} \psi_j$, with ψ_j (and $\bar{\psi}_j$) Grassmann independent fields, $P(d\psi_j)$ a Gaussian fermionic measure with covariance $g_j(x-y)$. Analagously, $P(d\psi_{(\geqslant j)})$ will indicate the Gaussian measure with covariance $g_{(\geqslant j)}(x-y) = \sum_{k=j}^{\infty} g_k(x-y)$ acting on $\psi_{(\geqslant j)} = \sum_{k=j}^{\infty} \psi_k$. Note the parallel between the smooth and the block formalism: the jth scale fields ψ_j and $\bar{\psi}_j$ in (2.10) correspond to the fields $M_j Q \zeta_j$ and $\bar{\zeta}_j Q^\dagger M_j^\dagger$ in (3.11) in [PP2]; and the massive interaction in the jth scale expressed in the smooth RG by the measure $P(d\psi_j)$, with covariance g_j, in the block RG corresponds to the integration with covariance $\tilde{\Gamma}_j$, i.e.,

$$\int P(d\psi_j) \, f(\bar{\psi}_j, \psi_j) \leftrightarrow \frac{\int d\bar{\zeta}_j \, d\zeta_j \exp[-\bar{\zeta}_j Q^\dagger D_j Q \zeta_j] \, f(\bar{\zeta}_j Q^\dagger M_j^\dagger, M_j Q \zeta_j)}{(\text{numerator with } f = 1)}$$

We remark that, as $g_j(0) = 0$, everything goes as if $[\bar{\psi}\psi]^2$ (in the effective potential) were Wick ordered.

Now we derive the correlation function formulas using this multiscale decomposition. We can start with the more regularized expressions above (with the cutoff M) and take the limit $M \to \infty$ later. But, as in our further analysis the bounds will be uniform on M, and we will drop M from the notation.

The generating function is given by

$$Z(h, \bar{h}) = \int P_{b_0}(d\psi) \exp[-V_0(\psi, \bar{\psi}) + \bar{h}\psi + \bar{\psi}h] \tag{2.11}$$

$\psi = \psi_{(\geqslant 0)}$, $P_{b_0}(d\psi)$ meaning a Gaussian fermionic measure with covariance $b_0^{-1} g_{(\geqslant 0)}(x-y)$. Now we follow the same procedures adopted to construct the block formalism in [PP2]. We start the first RG step writing $\psi_{(\geqslant 0)} = \psi_{(\geqslant 1)} + \psi_0$ and $P_{b_0}(d\psi_{(\geqslant 0)}) = P_{b_0}(d\psi_{(\geqslant 1)})\, P_{b_0}(d\psi_0)$, and so,

$$Z(h, \bar{h}) = c \int P_{b_0}(d\psi_{(\geqslant 1)})\, P_{b_0}(d\psi_0) \exp[\bar{h}\psi_{(\geqslant 1)} + \bar{\psi}_{(\geqslant 1)} h + \bar{h}\psi_0 + \bar{\psi}_0 h]$$
$$\times \exp[-V_0(\psi_{(\geqslant 1)} + \psi_0, \ldots)] \tag{2.12}$$

With the shift $\psi_0 \to \psi_0 + b_0^{-1} g_0 h$, $\bar{\psi}_0 \to \bar{\psi}_0 + b_0^{-1} \bar{h} g_0$, we get

$$Z(h, \bar{h}) = \exp[b_0^{-1} \bar{h} g_0 h]\, c \int P_{b_0}(d\psi_{(\geqslant 1)})\, P_{b_0}(d\psi_0) \exp[\bar{h}\psi_{(\geqslant 1)} + \bar{\psi}_{(\geqslant 1)} h]$$
$$\times \exp[-V_0(\psi_{(\geqslant 1)} + \psi_0 + b_0^{-1} g_0 h, \ldots)]$$

where $\bar{h} g_0 h$ means $\int dx\, dy\, \bar{h}(x)\, g_0(x-y)\, h(y)$. After defining V_1 as

$$\exp[-V_1(\psi_{(\geqslant 1)} + b_0^{-1} g_0 h, \bar{\psi}_{(\geqslant 1)} + b_0^{-1} \bar{h} g_0)]$$
$$= \int P_{b_0}(d\psi_0) \exp[-V_0(\psi_{(\geqslant 1)} + \psi_0 + b_0^{-1} g_0 h, \ldots)] \tag{2.13}$$

we extract the marginal quadratic term

$$V_1(\chi, \bar{\chi}) = \tilde{V}_1(\chi, \bar{\chi}) + \delta b_0 \bar{\chi}\, D\chi \tag{2.14}$$

with $\chi = \psi_{(\geqslant 1)} + b_0^{-1} g_0 h$, $D = i\not{\partial}$. Hence,

$$\begin{aligned}\delta b_0 \bar{\chi}\, D\chi &= \delta b_0 [(\bar{\psi}_{(\geqslant 1)} + b_0^{-1} \bar{h} g_0)\, D(\psi_{(\geqslant 1)} + b_0^{-1} g_0 h)] \\ &= \delta b_0 \bar{\psi}_{(\geqslant 1)}\, D\psi_{(\geqslant 1)} + \delta b_0 b_0^{-2} \bar{h} g_0\, D g_0 h \\ &\quad + \delta b_0 b_0^{-1} \bar{\psi}_{(\geqslant 1)}\, D g_0 h + \delta b_0 b_0^{-1} \bar{h} g_0\, D\psi_{(\geqslant 1)}\end{aligned}$$

If we had the orthogonality property (as in the block RG) we would get the simplifications

$$g_0\, D g_0 = g_0, \qquad \bar{\psi}_{(\geqslant 1)}\, D g_0 h = \bar{h} g_0\, D\psi_{(\geqslant 1)} = 0$$

Anyway, the generating functions becomes

$$Z(h, \bar{h}) = \exp[b_0^{-1} \bar{h}(g_0 - \delta b_0 b_0^{-1} g_0 \, D g_0) \, h]$$
$$\times \int P_{b_0}(d\psi_{(\geqslant 1)}) \exp[-\delta b_0 \bar{\psi}_{(\geqslant 1)} \, D\psi_{(\geqslant 1)}]$$
$$\times \exp[\bar{h}(1 - \delta b_0 b_0^{-1} g_0 \, D) \, \psi_{(\geqslant 1)} + \bar{\psi}_{(\geqslant 1)}(1 - \delta b_0 b_0^{-1} \, D g_0) \, h]$$
$$\times \exp[-\tilde{V}_1(\psi_{(\geqslant 1)} + b_0^{-1} g_0 h, \ldots)]$$

We define (note that g_0 and D are odd functions, translational invariant, and so $g_0 D = D g_0$)

$$Q_0 = 1, \qquad Q_1 = 1 - \delta b_0 b_0^{-1} g_0 \, D, \qquad G_1 = b_0^{-1} g_0,$$
$$P_1 = b_0^{-1} g_0 - \delta b_0 b_0^{-2} g_0 \, D g_0 = G_1 Q_1 \tag{2.15}$$

In the case with the orthogonality property [PP2], $P_1 = b_0^{-1} g_0 - \delta b_0 b_0^{-2} g_0 = \gamma_1^{(1)} g_0$, since (for such case) $g_0 \, D g_0 = g_0$. There, the operators G_n and P_n have a similar structure, written as a sum of the propagators Γ_j (however, with different coefficients). Here, such formulas will not be possible, but we still try to make explicit G_n and write P_n in terms of it. Another delicate point is the wavefunction renormalization. In the block RG formalism this process is automatic [PP2]: $\delta b_k \bar{\chi} \, D\chi$ automatically gives $\delta b_k \bar{\xi}_k \, D_k \xi_k$ (ξ_k, $\bar{\xi}_k$ are the fields at scale k, and D_k the effective free action). But, unfortunately, it does not happen now. That is, we cannot write $P_{b_0}(d\psi_{(\geqslant 1)}) \exp[-\delta b_0 \bar{\psi}_{(\geqslant 1)} \, D\psi_{(\geqslant 1)}]$ as $P_{b_1}(d\psi_{(\geqslant 2)}) \, P_{b_1}(d\psi_1)$. However, as described in [BGPS], we may write

$$P_{b_0}(d\psi_{(\geqslant 1)}) \exp[-\delta b_0 \bar{\psi}_{(\geqslant 1)} \, D\psi_{(\geqslant 1)}] = N' P_{b_1}(d\psi_{(\geqslant 2)}) \, \tilde{P}_{b_1}(d\psi_1)$$

where $\tilde{P}_{b_1}$, calculated below, behaves (for long distances) as P_{b_1}. Let us show it. For the Fourier transform of the covariance $[b_0 i\partial\!\!\!/_{(\geqslant 1)} + \delta b_0 i\partial\!\!\!/]^{-1}$ we have $[b_0 \exp[L^2 p^2] + \delta b_0]^{-1} \, p\!\!\!/ / p^2$ and

$$[b_0 e^{L^2 p^2} + \delta b_0]^{-1}$$
$$= \frac{1}{b_1} e^{-L^4 p^2} + \frac{1}{b_1} [e^{-L^2 p^2} - e^{-L^4 p^2}] + \frac{1}{b_1} \delta b_0 \left[\frac{e^{-L^2 p^2} - e^{-2L^2 p^2}}{b_0 + \delta b_0 e^{-L^2 p^2}} \right] \tag{2.16}$$

where $b_1 = b_0 + \delta b_0$. Hence,

$$[b_0 i\partial\!\!\!/_{(\geqslant 1)} + \delta b_0 i\partial\!\!\!/]^{-1} \, (x - y)$$
$$= \frac{1}{b_1} \, g_{(\geqslant 2)}(x - y) + \frac{1}{b_1} \, [g_1(x - y) + r_1(x - y)] \tag{2.17}$$

where

$$r_1(x-y) = \delta b_0 \int \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, \frac{\not{p}}{p^2}\, \frac{e^{-L^2p^2} - e^{-2L^2p^2}}{b_0 + \delta b_0 e^{-L^2p^2}} \tag{2.18}$$

and, defining $\tilde{g}_1(x-y) = g_1(x-y) + r_1(x-y)$, we still keep the bound

$$|\tilde{g}_1(x-y)| \leqslant \text{const.}\ L^{-2 \cdot 1} \exp[-\beta L^{-1} |x-y|] \tag{2.19}$$

($\beta > 0$) if δb_0 small enough (to avoid a pole in the expression of r_1 above; in fact, for the model to be analyzed here we show in [PP1] that $\delta b_0 = \mathcal{O}(\lambda^2)$).

Thus, returning to the generating function, we have

$$Z(h, \bar{h}) = \exp[\bar{h} P_1 h]\ c \int P_{b_1}(d\psi_{(\geqslant 2)})\ \tilde{P}_{b_1}(d\psi_1) \\ \times \exp[\bar{h} Q_1 \psi_{(\geqslant 1)} + \bar{\psi}_{(\geqslant 1)} Q_1 h] \exp[-\tilde{V}_1(\psi_{(\geqslant 1)} + G_1 h,...)] \tag{2.20}$$

To make clear the representation, let us perform also the second RG step in details. We start, again, separating ψ_1 and making the shift $\psi_1 \to \psi_1 + b_1^{-1} \tilde{g}_1 Q_1 h$ (similarly for $\bar{\psi}_1$). Hence,

$$\begin{aligned} Z(h, \bar{h}) = \exp[\bar{h} P_1 h]\ c \int P_{b_1}(d\psi_{(\geqslant 2)})\ \tilde{P}_{b_1}(d\psi_1) \\ \times \exp[b_1^{-1} \bar{h} Q_1 \tilde{g}_1 Q_1 h + \bar{h} Q_1 \psi_{(\geqslant 2)} + \bar{\psi}_{(\geqslant 2)} Q_1 h] \\ \times \exp[-\tilde{V}_1(\psi_{(\geqslant 2)} + G_1 h + b_1^{-1} \tilde{g}_1 Q_1 h + \psi_1,...)] \end{aligned}$$

where we have used that $Q_1(x-y)$ is even. We define

$$G_2 = G_1 + b_1^{-1} \tilde{g}_1 Q_1 \tag{2.21}$$

$$\exp[-V_2(\psi_{(\geqslant 2)} + G_2 h,...)] = \int \tilde{P}_{b_1}(d\psi_1) \exp[-\tilde{V}_1(\psi_{(\geqslant 2)} + \psi_1 + G_2 h,...)] \tag{2.22}$$

and separate in V_2 the marginal quadratic part $V_2(\chi, \bar{\chi}) = \delta b_1 \bar{\chi} D\chi + \tilde{V}_2(\chi, \bar{\chi})$, with $\chi = \psi_{(\geqslant 2)} + G_2 h$, $\bar{\chi} = \bar{\psi}_{(\geqslant 2)} + \bar{h} G_2$,

$$\begin{aligned} \bar{\chi} D\chi &= (\bar{\psi}_{(\geqslant 2)} + \bar{h} G_2)\ D(\psi_{(\geqslant 2)} + G_2 h) \\ &= \bar{\psi}_{(\geqslant 2)}\ D\psi_{(\geqslant 2)} + \bar{h} G_2\ D G_2 h + \bar{h} G_2\ D\psi_{(\geqslant 2)} + \bar{\psi}_{(\geqslant 2)}\ D G_2 h \end{aligned}$$

(for the block RG, the crossed terms above vanish and $G_2 DG_2$ does not mix the scales, i.e., it is written as a sum of g_0 and $\tilde{g}_1$ without a product term). The generating function becomes

$$Z(h, \bar{h}) = \exp[\bar{h}(P_1 + b_1^{-1} Q_1 \tilde{g}_1 Q_1 - \delta b_1 G_2\, DG_2)\, h]$$
$$\times c \int P_{b_1}(d\psi_{(\geqslant 2)}) \exp[-\delta b_1 \bar{\psi}_{(\geqslant 2)}\, D\psi_{(\geqslant 2)} + \bar{h} Q_1 \psi_{(\geqslant 2)} + \bar{\psi}_{(\geqslant 2)} Q_1 h]$$
$$\times \exp[-\delta b_1 \bar{h} G_2\, D\psi_{(\geqslant 2)} - \delta b_1 \psi_{(\geqslant 2)}\, DG_2 h]$$
$$\times \exp[-\tilde{V}_2(\psi_{(\geqslant 2)} + G_2 h, ...)]$$

Let us define

$$Q_2 = Q_1 - \delta b_1 G_2\, D, \qquad P_2 = P_1 + b_1^{-1} Q_1 \tilde{g}_1 Q_1 - \delta b_1 G_2\, DG_2 \tag{2.23}$$

Thus,

$$P_2 = G_2 Q_1 - \delta b_1 G_2\, DG_2 = G_2 Q_2 \tag{2.24}$$

$$Z(h, \bar{h}) = \exp[\bar{h} P_2 h]\; c \int P_{b_1}(d\psi_{(\geqslant 2)}) \exp[-\delta b_1 \bar{\psi}_{(\geqslant 2)}\, D\psi_{(\geqslant 2)}]$$
$$\times \exp[\bar{h} Q_2 \psi_{(\geqslant 2)} + \bar{\psi}_{(\geqslant 2)} Q_2 h] \exp[-\tilde{V}_2(\psi_{(\geqslant 2)} + G_2 h, ...)]$$

Now we renormalize the wavefunction term

$$P_{b_1}(d\psi_{(\geqslant 2)}) \exp[-\delta b_1 \bar{\psi}_{(\geqslant 2)}\, D\psi_{(\geqslant 2)}] = N'' P_{b_2}(d\psi_{(\geqslant 3)})\, \tilde{P}_{b_2}(d\psi_2)$$

where N'' is the normalization factor, $b_2 = b_1 + \delta b_1$, $\tilde{P}_{b_2}$, with covariance $b_2^{-1} \tilde{g}_2$ given by

$$\tilde{g}_2(x-y) = g_2(x-y) + r_2(x-y)$$
$$r_2(x-y) = \delta b_1 \int \frac{d^3 p}{(2\pi)^3}\, e^{ip(x-y)}\, \frac{\not{p}}{p^2} \frac{e^{-L^4 p^2} - e^{-2L^4 p^2}}{b_1 + \delta b_1 e^{-L^4 p^2}} \tag{2.25}$$

with

$$|\tilde{g}_2(x-y)| \leqslant \text{const. } L^{-2 \cdot 2} \exp[-\beta L^{-2}\, |x-y|] \tag{2.26}$$

δb_1 small enough. Hence,

$$Z(h, \bar{h}) = \exp[\bar{h} P_2 h]\; N_2 \int P_{b_2}(d\psi_{(\geqslant 3)})\, \tilde{P}_{b_2}(d\psi_2)$$
$$\times \exp[\bar{h} Q_2 \psi_{(\geqslant 2)} + \bar{\psi}_{(\geqslant 2)} Q_2 h] \exp[-\tilde{V}_2(\psi_{(\geqslant 2)} + G_2 h, ...)]$$

Iterating, after n steps,

$$Z(h,\bar{h}) = \exp[\bar{h}P'_n h]\, N_n \int P_{b_{n-1}}(d\psi_{(\geqslant n)}) \\ \times \exp[\bar{h}Q_{n-1}\psi_{(\geqslant n)} + \bar{\psi}_{(\geqslant n)}Q_{n-1}h]\exp[-V_n(\psi_{(\geqslant n)} + G_n h,...)] \tag{2.27}$$

where

$$\exp[-V_n(\psi_{(\geqslant n)} + G_n h,...)] \\ = \int \tilde{P}_{b_{n-1}}(d\psi_{n-1}) \exp[-\tilde{V}_{n-1}(\psi_{(\geqslant n)} + \psi_{n-1} + G_n h,...)]$$

$$G_n = G_{n-1} + b_{n-1}^{-1}\, \tilde{g}_{n-1} Q_{n-1}, \qquad Q_{n-1} = Q_{n-2} - \delta b_{n-2} G_{n-1}\, D$$

$$P'_n = P_{n-1} + b_{n-1}^{-1} Q_{n-1}\, \tilde{g}_{n-1} Q_{n-1}, \qquad P_{n-1} = G_{n-1} Q_{n-1}$$

$$\tilde{g}_{n-1}(x-y) = g_{n-1}(x-y) + r_{n-1}(x-y) \tag{2.28}$$

$$r_{n-1}(x-y) = \delta b_{n-2} \int \frac{d^3p}{(2\pi)^3}\, e^{ip(x-y)}\, \frac{\not{p}}{p^2}\, \frac{e^{-L^{2(n-1)}p^2} - e^{-2L^{2(n-1)}p^2}}{b_{n-2} + \delta b_{n-2} e^{-L^{2(n-1)}p^2}} \tag{2.29}$$

$$|\tilde{g}_{n-1}(x-y)| \leqslant \text{const.}\, L^{-2\cdot(n-1)} \exp[-\beta L^{-(n-1)}\, |x-y|] \tag{2.30}$$

V_n above still contains a marginal term in which shall renormalize the wavefunction (contributing with δb_{n-1}). The kernels of V_n do not present singularities, and so, the pointwise bounds (to be used later) appear without further adjustments (more comments in [PP1]). Thus, we keep V_n in the final step of the generating formula. However, note that for $n \to \infty$ the difference between V_n and $\tilde{V}_n$ disappears; note also that (following the formalism construction) V_n depends only on $\tilde{V}_{n-1}$.

Performing the final shift

$$\psi_{(\geqslant n)} \to \psi_{(\geqslant n)} + b_{n-1}^{-1}\, \tilde{g}_{(\geqslant n)} Q_{n-1} h, \qquad \bar{\psi}_{(\geqslant n)} \to \bar{\psi}_{(\geqslant n)} + b_{n-1}^{-1} \bar{h} Q_{n-1} \tilde{g}_{(\geqslant n)} \tag{2.31}$$

where $\tilde{g}_{(\geqslant n)} = \tilde{g}_n + g_{(\geqslant n+1)}$, we get

Lemma 2.1. For the "continuous" fermionic generating function (2.11) we have

$$Z(h,\bar{h}) = \exp[\bar{h}\tilde{P}_n h]\, N_n \int P_{b_{n-1}}(d\psi_{(\geqslant n)}) \\ \times \exp[-V_n(\psi_{(\geqslant n)} + \tilde{G}_n h, \bar{\psi}_{(\geqslant n)} + \bar{h}\tilde{G}_n)] \tag{2.32}$$

with

$$\tilde{P}_n = P'_n + b_{n-1}^{-1} Q_{n-1} \tilde{g}_{(\geqslant n)} Q_{n-1}, \qquad \tilde{G}_n = G_n + b_{n-1}^{-1} \tilde{g}_{(\geqslant n)} Q_{n-1} \tag{2.33}$$

where n is the number of "smooth" RG steps, and P'_n, Q_n and G_n are given by Eqs. (2.15), (2.28).

Thus, the truncated correlation functions easily follow (once more, we supress the spinor indices: the bounds to be presented are valid for any set of indices α)

$$\begin{aligned} &S_{2k}(x_1, x_2,..., x_{2k}) \\ &\quad = \delta_{1,k} \tilde{P}_n(x_1, x_2) - \int dy_1,..., dy_{2k} \prod_{i=1}^{2k} [\tilde{G}_n(x_i, y_i)] \\ &\qquad \times \frac{\partial}{\partial \bar{\chi}_k} \cdots \frac{\partial}{\partial \bar{\chi}_1} W_n \frac{\partial}{\partial \chi_{2k}} \cdots \frac{\partial}{\partial \chi_{k+1}} \bigg|_{\bar{\chi}_1,..., \bar{\chi}_k, \chi_{k+1},..., \chi_{2k} = 0} (y_1,..., y_{2k}) \end{aligned} \tag{2.34}$$

where

$$\exp[-W_n(\chi, \bar{\chi})] = \frac{\int P_{b_{n-1}}(d\psi_{(\geqslant n)}) \exp[-V_n(\psi_{(\geqslant n)} + \chi, \bar{\psi}_{(\geqslant n)} + \bar{\chi})]}{\text{numerator with } \chi, \bar{\chi} = 0}$$

3. THE TRUNCATED CORRELATION FUNCTION BOUNDS AND TREE DECAY

In this section, using the obtained multiscale formalism, we study the k-point truncated function of the tridimensional Gross–Neveu model. Generally, in similar problems [BGPS, BO, FMRS, FMRT, GK4, IM, MS] just the two and four-point function are completely controlled (concerning a detailed decay analysis, etc), with comments (integral bounds, etc) about the general correlations. Here, as a first description, we obtain bounds which may be improved (still considering the formulas and initial bounds described), however, the whole description and the final bounds are already precise enough to show the long distance tree decay for the truncated correlation functions (and also analyticity on λ, the initial potential strength, etc).

A delicate point now is the bound on the effective potential: we need a pointwise bound for the kernel $K_n^{m,a}(y_1,..., y_k)$ (the notation means the kernel of V_n with $2m$ fields and a derivatives, at points $y_1,..., y_k$). A detailed study of the effective potential flow for the tridimensional Gross–Neveu model within the smooth RG approach has been carried out (using the

Gallavotti Nicoló tree expansion) by two of us in a previous paper [PP1]. We proved the following theorems related to the kernels of the effective potential at scale n and the flow of the running coupling constants.

Theorem 3.1. For small λ ($|\lambda| \leqslant \varepsilon$), the effective potential at scale n, V_n, can be written as

$$
\begin{aligned}
V_n(\psi, \bar{\psi}) = {} & \lambda \int_\Lambda dy (\bar{\psi}_y \psi_y)^2 + \int_\Lambda dy_1 \int_\Lambda dy_2 K_n^{2,0}(y_1 - y_2)\, \bar{\psi}_{y_1} \psi_{y_2} \\
& + \int_\Lambda dy_1 \int_\Lambda dy_2\, K_n^{2,1}(y_1 - y_2)\, \bar{\psi}_{y_1}\, \partial^2 \psi_{y_2} \\
& + \sum_{m=2}^{\infty} \sum_{a=0}^{m} \sum_{p=\max\{m-1+a,\, 2\}}^{2m} \int_\Lambda dy_1 \cdots \int_\Lambda dy_p \\
& \times \sum_{r_1,\dots,r_{p-a}} \sum_{s_1,\dots,s_{p-a}} K_n^{m,a,\{r_j,s_j\}}(y_1,\dots,y_p) \\
& \times \bar{\psi}_{y_1}^{r_1} \cdots \bar{\psi}_{y_{p-a}}^{r_{p-a}} \psi_{y_1}^{s_1} \cdots \psi_{y_{p-a}}^{s_{p-a}}\, \partial^2 \psi_{y_{p-a+1}} \cdots \partial^2 \psi_{y_p}
\end{aligned}
\tag{3.1}
$$

where, for $j = 1, 2,..., p-a$, $0 \leqslant r_j \leqslant 2$. The same for s_j. For $j = 1, 2,..., p-a$, we have $1 \leqslant r_j + s_j \leqslant 3$ and $\sum_j r_j = m$, $\sum_j s_j = m - a$.

The kernels $K_n^{2,0}$, $K_n^{2,1}$, $K_n^{m,a,\{r_j,s_j\}}$ are analytic in λ (uniformly in Λ) and admit the pointwise bound

$$
\begin{aligned}
& |K_n^{m,a}(y_1,\dots,y_p)| \\
& \quad = \left| \sum_{\{r_j,s_j\}} K_n^{m,a,\{r_j,s_j\}}(y_1,\dots,y_p) \right| \\
& \quad \leqslant (c\,|\lambda|)^p \sum_{\tau \in \{1,2,\dots,p\}} B_\tau \sum_{j_1=0}^{n-1} \cdots \sum_{j_{p-1}=0}^{n-1} \sum_{\alpha_1,\dots,\alpha_{p-1}} L^{-\alpha_1 j_1} \cdots L^{-\alpha_{p-1} j_{p-1}} \\
& \quad\quad \times e^{-\beta L^{-j_1} |\ell_1|} \cdots e^{-\beta L^{-j_{p-1}} |\ell_{p-1}|}
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
& |K_n^{2,0}(y_1 - y_2)| \\
& \quad \leqslant c\lambda^2 L^{-n} \sum_{j=0}^{n-1} L^{-6j} \exp[-\beta L^{-j} |y_1 - y_2|]
\end{aligned}
\tag{3.3}
$$

$$
\begin{aligned}
& |K_n^{2,1}(y_1 - y_2)| \\
& \quad \leqslant c\lambda^2 \sum_{j=0}^{n-1} L^{-4j} \exp[-\beta L^{-j} |y_1 - y_2|]
\end{aligned}
\tag{3.4}
$$

where c is a constant; the sum $\sum_{\alpha_1,..., \alpha_{p-1}}$ is over positive integer values of α_j such that, for $d=3$,

$$\sum_{j=1}^{p-1} \alpha_j = (d+1)\, p - [2m(d-1)/2 + 2a] \tag{3.5}$$

the sum $\sum_{\tau \in \{1, 2,..., p\}}$ is over all connected tree graphs between $y_1, y_2,..., y_p$ (thus with $p-1$ branches $\ell_1,..., \ell_{p-1}$, with $\ell_i = y_{i_1} - y_{i_2}$); and B_τ are combinatorial factors such that $\sum_\tau B_\tau \leqslant c^p$.

Again, we supressed the spinor indices (the bounds follow for any set of these indices).

Theorem 3.2. The wavefunction renormalization δb_n is analytic in λ (for small λ, $|\lambda| \leqslant \varepsilon$, ε a positive constant) and

$$|\delta b_n| \leqslant cnL^{-2n}\lambda^2 \tag{3.6}$$

where c is a positive constant.

Let us make some remarks.

1. The effective potential described in Theorem 3.1 (expressions to be used in the correlation formulas) does not vanish as $n \to \infty$: in the case of the block RG approach where everything is more transparent, this potential describes the interaction in the unit lattice, and only after rescaled to "thin" lattices of spacement $1/L^n$ they shrink to zero (due to scaling factors introducing negative powers of L^n).

2. For the block RG approach [PP2], starting with an action with chiral symmetry (in order to prevent the generation of the relevant massive term), after dealing with the technical problems due to the lack of translational invariance, a similar theorem can be established: in the lattice, the free propagator decomposition gives $\tilde{\Gamma}_0, \tilde{\Gamma}_1,..., \tilde{\Gamma}_k,...$, with the same bounds as $\tilde{g}_j$, but with $|\partial \tilde{\Gamma}_j \partial^\dagger(x, y)| \leqslant cL^{-(2-\varepsilon)\, j} \exp[-\beta L^{-j}\, |x-y|]$, a factor $L^{j\varepsilon}$ worse than $|\partial \partial \tilde{g}_j|$, which slightly changes the sum on α_j (3.5) with the replacement of $d+1$ by $d+1-\varepsilon$, and also change the bound (3.6) with the replacement of $2n$ by $(2-\varepsilon)\, n$.

Now, with the bounds obtained from the effective potential study, we turn to the correlation formulas. First we analyze the propagators P_n and G_n (the difference between $\tilde{P}_n$, $\tilde{G}_n$ and P_n, G_n is a correction which vanishes as $n \to \infty$, thus, it is enough to consider P_n and G_n).

As we know from (3.12) in [PP2], for the block RG approach,

$$P_n = \sum_{j=0}^{n-1} \gamma_j^{(n)} \tilde{\Gamma}_j, \qquad \gamma_j^{(n)} = b_j^{-1} - (b_n - b_j)\, b_j^{-2}$$
$$|\tilde{\Gamma}_j(x, y)| \leqslant cL^{-j(d-1)} \exp[-\beta L^{-j} |x-y|]$$

We write, as $n \to \infty$,

$$P_\infty = \sum_{j=0}^{\infty} \gamma_j^{(\infty)} \tilde{\Gamma}_j = b_\infty^{-1} \sum_{j=0}^{\infty} \tilde{\Gamma}_j + \sum_{j=0}^{\infty} (\gamma_j^{(\infty)} - b_\infty^{-1})\, \tilde{\Gamma}_j \equiv b_\infty^{-1} D^{-1} + \mathscr{C} \tag{3.7}$$

But $b_\infty = b_0 + \sum_{j=0}^{\infty} \delta b_j \Rightarrow$ (from Theorem 3.2 above) $|b_\infty| \leqslant |b_0| + c\lambda^2 \times \sum_{j=0}^{\infty} L^{-j(2-\varepsilon)} = |b_0| + \mathscr{O}(\lambda^2)$, and $\gamma_j^{(\infty)} - b_\infty^{-1} = b_j^{-1} - (b_\infty - b_j)\, b_j^{-2} - b_\infty^{-1} = -b_\infty^{-1}([b_\infty - b_j]/b_j)^2$. Thus,

$$|\mathscr{C}(x, y)| \leqslant cb_\infty^{-2} \sum_{j=0}^{\infty} L^{-(2-\varepsilon)\, j} L^{-j(d-1)} \exp[-\beta L^{-j} |x-y|]$$
$$\leqslant cb_\infty^{-2}/(1+|x-y|)^{d+1-\varepsilon} \tag{3.8}$$

that is, subdominant in relation to D^{-1}. For G_n we have

$$G_n = b_n^{-1} \sum_{j=0}^{n-1} \tilde{\Gamma}_j + \sum_{j=0}^{n-1} (b_j^{-1} - b_n^{-1})\, \tilde{\Gamma}_j$$

and so, as $n \to \infty$,

$$G_\infty(x, y) = b_\infty^{-1} D^{-1}(x, y) + \sum_{j=0}^{\infty} (b_j^{-1} - b_\infty^{-1})\, \tilde{\Gamma}_j$$
$$\left| \sum_{j=0}^{\infty} (b_j^{-1} - b_\infty^{-1})\, \tilde{\Gamma}_j \right| \leqslant cb_\infty^{-2} \sum_{j=0}^{\infty} L^{-(2-\varepsilon)\, j} L^{-j(d-1)} \exp[-\beta L^{-j} |x-y|]$$
$$\leqslant cb_\infty^{-2}/(1+|x-y|)^{d+1-\varepsilon} \tag{3.9}$$

That is, $|G_\infty(x, y)| \leqslant b_\infty^{-1}/(1+|x-y|)^{d-1} + c/(1+|x-y|)^{d+1-\varepsilon}$.

For the smooth RG approach the analysis is more elaborate (due to the absence of the orthogonality property, which leads to a mix between scales), but we still obtain

$$\begin{aligned} G_\infty(x, y) &= b_\infty^{-1} D^{-1} + \mathscr{C}_1(x, y), \qquad |\mathscr{C}_1(x, y)| \leqslant c/(1+|x-y|)^{d+1-\varepsilon} \\ P_\infty(x, y) &= b_\infty^{-1} D^{-1} + \mathscr{C}_2(x, y), \qquad |\mathscr{C}_2(x, y)| \leqslant c/(1+|x-y|)^{d+1-\varepsilon} \end{aligned} \tag{3.10}$$

Let us show it. According to our previous definitions (2.28)

$$P_n = G_n Q_n, \qquad G_n = G_{n-1} + b_{n-1}^{-1} \tilde{g}_{n-1} Q_{n-1}, \qquad G_0 = 0, \qquad G_0 = 1$$
$$Q_n = Q_{n-1} - \delta b_{n-1}\, DG_n, \qquad b_n = b_{n-1} + \delta b_{n-1}$$

From these formulas, we may write

$$G_n = \sum_{j=0}^{n-1} b_j^{-1} \tilde{g}_j + \sum_{j=0}^{n-1} \rho_j, \qquad Q_n = 1 + \sum_{j=0}^{n-1} Dq_j^{(n)} \tag{3.11}$$

(where $\rho_0 = 0$), which iterates with

$$\rho_n = b_n^{-1} \sum_{j=0}^{n-1} \tilde{g}_n\, Dq^{(n)}{}_j$$
$$Dq_j^{(n+1)} = Dq_j^{(n)} - \delta b_n b_j^{-1}\, D\tilde{g}_j - \delta b_n\, D\rho_j, \qquad j < n \tag{3.12}$$
$$Dq_n^{(n+1)} = -\delta b_n b_n^{-1}\, D\tilde{g}_n - \delta b_n\, D\rho_n$$

($\rho_0 = 0 \Rightarrow Dq_0^{(1)} = -\delta b_0 b_0^{-1}\, D\tilde{g}_0$). Making explicit all the scales in the expressions above, one can see that ρ_n involves terms (mixing the scales) such as $\tilde{g}_1\, D\tilde{g}_0$, $\tilde{g}_2\, D\tilde{g}_1\, D\tilde{g}_0$, etc., and $q_j^{(n)}$ terms as $D\tilde{g}_0$, $D\tilde{g}_1\, D\tilde{g}_0$, etc. Hence, we have to know how to control $g_k\, Dg_j$, $k > j$ (since $\tilde{g}_k$ is essentially given by g_k), and related expressions. From the formula (2.6) for g_j we have

$$g_k\, Dg_j(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{(e^{-L^{2k}p^2} - e^{-L^{2k+2}p^2})}{p^2}\, \not{p}\not{p}\, \frac{(e^{-L^{2j}p^2} - e^{-L^{2j+2}p^2})}{p^2}\, \not{p} e^{ip(x-y)}$$

and making the change of variables $L^k p \to p$

$$g_k\, Dg_j(x-y) = L^{-2k} L^{-2(k-j)} (L^2 - 1) \int \frac{d^3p}{(2\pi)^3} (e^{-p^2} - e^{-L^2p^2})$$
$$\times \not{p} e^{-L^{-2(k-j)}p^2} f(p)\, e^{iL^{-k}p(x-y)}$$

with $f(p)$ dominated by $(e^{-p^2} - e^{-L^2p^2})$. Thus, it follows

$$|g_k\, Dg_j(x-y)| \leqslant L^{-2k} L^{-2(k-j)} L^2 c \exp[-\beta L^{-k}\, |x-y|]$$

With estimates like this, using $|\delta b_n| \leqslant \mathcal{O}(\lambda^2)\, nL^{-2n}$ ($\Rightarrow b_\infty = 1 + \mathcal{O}(\lambda^2)$), and the expressions (3.12) for ρ_n and $Dq_j^{(n)}$ above we obtain

$$|\rho_n(x-y)| \leqslant cL^{-(4-\varepsilon)\,n} \exp[-\beta L^{-n}\, |x-y|]$$
$$|Dq_j^{n+1}(x-y)| \leqslant c_{n+1} L^{-(5-\varepsilon)\,j} \exp[-\beta L^{-j}\, |x-y|] \tag{3.13}$$

(where $c_{n+1} < c$, for all n) which shows that G_n is dominated by the sum over $\tilde{g}_j$; Q_n by 1; and so, the formulas (3.10) for G_∞ and P_∞ above follow.

For the two point function, besides $\tilde{P}_n$ we still have

$$\int dy_1\, dy_2\, \tilde{G}_n(x_1, y_1)[\partial K_n^{2,1} \partial](y_1, y_2)\, \tilde{G}_n(y_2, x_2)$$

where $K_n^{2,1}$ is the kernel of the quadratic irrelevant part of V_n with two fields and two derivatives (that is why the term $\partial\partial$ appears above). Note that the term $K_n^{2,0}$ (see Theorem 3.1) contributes to the two-point function as $\mathcal{O}(L^{-n})\, D^{-1}(x_1, x_2)$, and so, vanishes as $n \to \infty$. Using Theorem 3.1, for the expression above, as $n \to \infty$, we get the bound (for $d = 3$, and in the smooth RG approach)

$$c \int dy_1\, dy_2 (1 + |x_1 - y_1|)^{-d} (1 + |y_1 - y_2|)^{-d-1} (1 + |y_2 - x_2|)^{-d}$$
$$< c(1 + |x_1 - x_2|)^{-d+\varepsilon}$$

where we used $|\partial G_\infty(x - y)| \leqslant 1/(1 + |x - y|)^d$. For the block RG approach, using the bounds obtained, we expect

$$c \sum_{y_1, y_2} (1 + |x_1 - y_1|)^{-d+\varepsilon'} (1 + |y_1 - y_2|)^{-d-1+\varepsilon} (1 + |y_2 - x_2|)^{-d+\varepsilon'}$$
$$< c(1 + |x_1 - x_2|)^{-d+\varepsilon''}$$

Now we argue to show that no more terms contribute, as $n \to \infty$, for the two point function (the same analysis follows for the k-point truncated correlations). In the smooth RG description (Lemma 2.1), the contribution due to the other parts of V_n with four or more fields keeps two (or more) fields $\psi_{(\geqslant n)}$ and $\bar{\psi}_{(\geqslant n)}$ inside the integral. The integration of such fields (note that, from $\psi_{(\geqslant n)}$, we need to separate ψ_n, with covariance $\tilde{g}_n$ and $\psi_{(\geqslant n+1)}$ with covariance $g_{(\geqslant n+1)}$) leads to extra terms such as $\tilde{g}_n$ and $g_{(\geqslant n+1)}$ which go to zero as n goes to infinity. A similar analysis follows for the block RG mechanism in [PP2].

Turning to the general correlations, for the k point truncated function $S_k(x_1,..., x_k)$ we have terms such as

$$\int dy_1 \cdots dy_p \prod_{i=1}^{k} [\tilde{G}_n(x_i, y_i)]\, K_n^{k,a}(y_1,..., y_p) \tag{3.14}$$

where $p \leqslant k$; p and a assuming the values allowed by Theorem 3.1. Now we use the previous bounds in order to control these expressions.

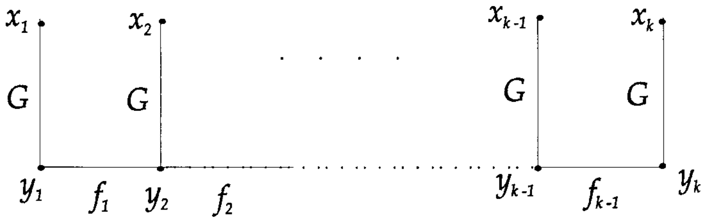

Fig. 1.

From Theorem 3.1, $K_n^{k,a}(y_1,..., y_p)$ is bounded in terms of trees connecting $y_1,..., y_p$. We first study the simplest tree involving k different points $y_1,..., y_k$, which contributes to $S_k(x_1,..., x_k)$ as in Fig. 1, where $G \equiv G_n(x, y)$, $f_i \equiv f_i(y_{i+1} - y_i)$ (f_i is the ith branch term in 3.2)

$$\begin{aligned} |G_n(x, y)| &\leqslant b_n^{-1} \sum_{j=0}^{n-1} L^{-2j} \exp[-\beta L^{-j} |x-y| \\ &\leqslant c(1+|x-y|)^{-2} \end{aligned} \tag{3.15}$$

$$\begin{aligned} |f_i(y_{i+1} - y_i)| &\leqslant c \sum_{j=0}^{n-1} L^{-j\alpha_i} \exp[-\beta L^{-j} |y_{i+1} - y_i| \\ &\leqslant c(1+|y_{i+1} - y_i|)^{-\alpha_i} \end{aligned} \tag{3.16}$$

We start the analysis manipulating the expression to "extract" a tree decay on $x_1,..., x_k$ (improvements may be obtained for the interested reader). Note that the first two points x_1 and x_2 are connected by links passing by y_1 and y_2 (Fig. 2) and that this part of the graph (i.e., the links $\{x_1, y_1\}\{y_1, y_2\}\{y_2, x_2\}$) is bounded by

$$\begin{aligned} \sum_{j_1, k_1, j_2} & L^{-j_1 2} L^{-k_1 \alpha_1} L^{-j_2 2} \exp\{-\beta L^{-j_1} |x_1 - y_1|\} \\ & \times \exp\{-\beta L^{-k_1} |y_1 - y_2|\} \exp\{-\beta L^{-j_2} |x_2 - y_2|\} \end{aligned}$$

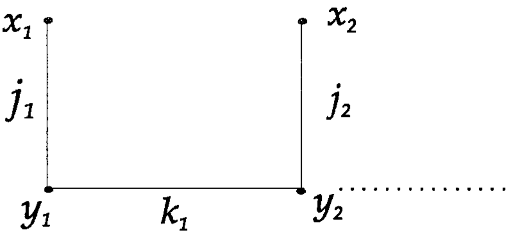

Fig. 2.

(times a constant). From the terms in the exponential, we separate a part (say, 1/4)

$$\frac{\beta}{4}\frac{|x_1-y_1|}{L^{j_1}}+\frac{\beta}{4}\frac{|y_1-y_2|}{L^{k_1}}+\frac{\beta}{4}\frac{|x_2-y_2|}{L^{j_2}}$$

and taking the largest index, say k_1, we write

$$\begin{aligned}&\frac{\beta}{4}\left[\frac{1}{L^{j_1}}-\frac{1}{L^{k_1}}\right]|x_1-y_1|+\frac{\beta}{4}\left[\frac{1}{L^{j_2}}-\frac{1}{L^{k_1}}\right]|x_2-y_2|\\&\qquad+\frac{\beta}{4}\frac{1}{L^{k_1}}\{|x_1-y_1|+|y_1-y_2|+|x_2-y_2|\}\\&\qquad\leqslant\frac{\beta}{4}\left[\frac{1}{L^{j_1}}-\frac{1}{L^{k_1}}\right]|x_1-y_1|+\frac{\beta}{4}\left[\frac{1}{L^{j_2}}-\frac{1}{L^{k_1}}\right]|x_2-y_2|+\frac{\beta}{4}\frac{|x_1-x_2|}{L^{k_1}}\end{aligned}$$

From the first three terms involving the powers on L, writing $L^{-k_1\alpha_1}=L^{-k_1(\alpha_1-\varepsilon)}L^{-k_1\varepsilon}$, we extract a factor involving L. Thus, after considering also the regions $j_1\geqslant k_1$, j_2 and $j_2\geqslant k_1$, j_1, simple manipulations lead us to the bound (for the links $\{x_1-\cdots-x_2\}$)

$$\begin{aligned}&3\sum_{\tilde{k}}L^{-\tilde{k}\varepsilon}\exp\left[-\frac{\beta}{4}\frac{|x_1-x_2|}{L^{\tilde{k}}}\right]\sum_{j_1,k_1,j_2}L^{-j_1(2-\varepsilon)}L^{-k_1(\alpha_1-\varepsilon)}L^{-j_2(2-\varepsilon)}\\&\qquad\times\exp\left[-\frac{3}{4}\beta L^{-j_1}|x_1-y_1|\right]\exp\left[-\frac{3}{4}\beta L^{-k_1}|y_1-y_2|\right]\\&\qquad\times\exp\left[-\frac{3}{4}\beta L^{-j_2}|x_2-y_2|\right]\end{aligned}$$

The first sum gives a term such as $c\,|x_1-x_2|^{-\varepsilon}$. Repeating the procedures for the link $\{x_2-x_3\}$ and following points we may bound the whole graph $\mathscr{G}$ by

$$\text{const}(1+|x_1-x_2|)^{-\varepsilon}(1+|x_2-x_3|)^{-\varepsilon}\cdots(1+|x_{k-1}-x_k|)^{-\varepsilon}\times\mathscr{G}' \tag{3.17}$$

where $\mathscr{G}'$ is given by $\mathscr{G}$ above with G_n replaced by G'_n (G'_n given by G_n with the factor 2 replaced by $2-\varepsilon$ and β by β', which is some fraction of β), and f_i replaced by f'_i (which is f_i with α_i replaced by $\alpha_i-\varepsilon$ and β by β'). We remark that the extraction procedure leads to a tree graph on $x_1,\ldots, x_k$ with the same structure of the initial graph on $y_1,\ldots, y_k$. For the case of graphs

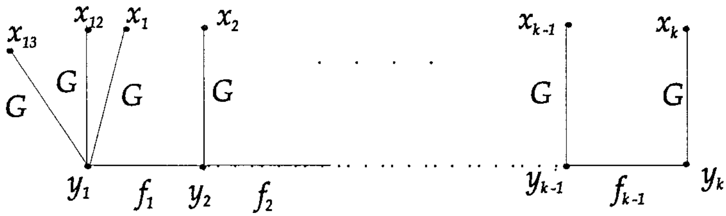

Fig. 3.

associated to kernels involving different fields at the same point, e.g., for $\mathscr{G}$ in Fig. 3, (i.e., the number of x larger than y), repeating the same procedures, we can still bound $\mathscr{G}$ by, e.g.,

$$\text{const.}\,[(1+|x_{12}-x_1|)^{-\varepsilon}(1+|x_{13}-x_1|)^{-\varepsilon}] \\ \times(1+|x_1-x_2|)^{-\varepsilon}(1+|x_2-x_3|)^{-\varepsilon}\cdots(1+|x_{k-1}-x_k|)^{-\varepsilon}\times\mathscr{G}'$$

again, with $\mathscr{G}'$ given by $\mathscr{G}$ with the replacements described above (2 by $2-\varepsilon$, β by β'). That is, the structure $y_1-y_2-\cdots$ is repeated in $x_1-x_2-\cdots$, and the extra fields x connected to y_1 are reflected as extra links in x_1.

Now we estimate $\mathscr{G}'$ for the simple tree of our first case. We shall bound $\mathscr{G}'(x_1=x_2=\cdots=x_k=0)\equiv\mathscr{G}'(0)$ (since $|\mathscr{G}'|\leqslant|c\mathscr{G}'(0)|$, with c not depending on x). We write

$$|\mathscr{G}'(0)|\leqslant\int\prod_{i=1}^{k-1}|[G'(y_i)\,f'_i(y_i-y_{i+1})]\,G'(y_k)|\,dy_1\cdots dy_k$$

$$=\int G'(y_1)(f'_1*h_1)(y_1)\,dy_1\leqslant\|G'\cdot(f'_1*h_1)\|_1 \tag{3.18}$$

where $h_1=G'\cdot[f'_2*[G'\cdot[f'_3*[G'\cdot[f'_4*\cdots]]]]]$.

From Hölder inequality (we drop, below, the prime out the notations of G' and f')

$$\|G\cdot(f_1*h_1)\|_1\leqslant\|G\|_s\,\|f_1*h_1\|_{s_1},\qquad s^{-1}+s_1^{-1}=1,\qquad 1\leqslant s_1,s\leqslant\infty \tag{3.19}$$

and from the Young inequality (see [RS])

$$\|f_1*h_1\|_{s_1}\leqslant\|f_1\|_{r_1}\,\|h_1\|_{p_1},\qquad p^{-1}+r_1^{-1}=1+s_1^{-1},\qquad 1\leqslant p_1,r_1,s_1\leqslant\infty \tag{3.20}$$

Hence,

$$\|G \cdot (f_1 * h_1)\|_1 \leqslant \|G\|_s \, \|f_1\|_{r_1} \, \|h_1\|_{p_1}, \qquad \text{with} \quad p^{-1} + r_1^{-1} = 2 - s^{-1}$$

For h_1,

$$\|h_1\|_{p_1} = \|G \cdot (f_2 * h_2)\|_{p_2} \leqslant \|G\|_s \, \|f_2 * h_2\|_{s_2}, \qquad p^{-1} = s^{-1} + s_2^{-1}$$

$$\|f_2 * h_2\|_{s_2} \leqslant \|f_2\|_{r_2} \, \|h_2\|_{p_2}, \qquad s_2^{-1} + 1 = r_2^{-1} + p_2^{-1}$$

And so,

$$|\mathscr{G}'(0)| \leqslant \|G\|_s \, \|f_1\|_{r_1} \, \|G\|_s \, \|f_2\|_{r_2} \, \|h_2\|_{p_2},$$
$$\text{with} \quad p_2^{-1} + r_2^{-1} + r_1^{-1} = 3 - 2s^{-1} \tag{3.21}$$

Iterating

$$|\mathscr{G}'(0)| \leqslant \|G\|_s \, \|f_1\|_{r_1} \, \|G\|_s \, \|f_2\|_{r_2} \cdots \|G\|_s \, \|f_{k-2}\|_{r_{k-2}} \, \|h_{k-2}\|_{p_{k-2}}$$
$$\text{with} \quad r_1^{-1} + r_2^{-1} + \cdots + r_{k-2}^{-1} + p_{k-2}^{-1} = k - 1 - (k-2)\, s^{-1} \tag{3.22}$$

But

$$\|h_{k-2}\|_{p_{k-2}} = \|G \cdot (f_{k-1} * G)\|_{p_{k-2}}$$
$$\leqslant \|G\|_s \, \|f_{k-1} * G\|_{s_{k-1}}, \qquad s^{-1} + s_{k-1}^{-1} = p_{k-2}^{-1}$$
$$\|f_{r-1} * G\|_{s_{k-1}} \leqslant \|f_{k-1}\|_{r_{k-1}} \, \|G\|_s, \qquad s^{-1} + r_{k-1}^{-1} = s_{k-1}^{-1} + 1$$

And so,

$$|\mathscr{G}'(0)| \leqslant \|G\|_s \, \|f_1\|_{r_1} \, \|G\|_s \, \|f_2\|_{r_2} \cdots \|G\|_s \, \|f_{k-1}\|_{r_{k-1}} \, \|G\|_s$$
$$\text{with} \quad r_1^{-1} + r_2^{-1} + \cdots + r_{k-1}^{-1} = k - ks^{-1} \tag{3.23}$$

To obtain a finite bound we need $s = \varepsilon' + 3/2$ (to get $\|G\|_s < \infty$) and $(\alpha_i - \varepsilon)\, r_i > 3$ (to get $\|f_i\|_{r_i} < \infty$). Thus, from (3.21) we have the following condition for the α's

$$\sum_{i=1}^{k-1} \alpha_i > (3 + \varepsilon'')\, k(1 - (\varepsilon' + 3/2)^{-1}) \geqslant k + \tilde{\varepsilon} \tag{3.24}$$

(ε', ε', $\tilde{\varepsilon} \ll 1$), which is easily satisfied for any graph with equal number of vertices and fields (see Theorem 3.1, Eq. (3.5)).

In order to generalize the bound (i.e., to control any tree), let us consider more complicated cases. The procedure for the extraction of the decay on x described above is completely general and follows for any graph (of course, with the adjustments described before for the graphs related to kernels with different fields at the same point). Thus, it is enough to bound $|\mathcal{G}'(0)|$.

First we analyze these graphs associated to kernels involving different fields at the same point, e.g., the graph of Fig. 3 above. Repeating the previous analysis,

$$|\mathcal{G}'(0)| \leqslant \int |[G'(y_1)]^3 f'_1(y_1 - y_2)\, G'(y_2)\, f'_2(y_2 - y_3) \cdots G'(y_k)|\, dy_1 \cdots dy_k$$
$$\leqslant \|G'^3 \cdot (f'_1 * h_1)\|_1 \tag{3.25}$$

and so (again, dropping the prime out the notations)

$$\|G^3 \cdot (f_1 * h_1)\|_1 \leqslant \|G^3\|_1\, \|f_1 * h_1\|_\infty$$

where we used that $|G^3|$ is integrable ($|G^2|$ is already integrable), and so, it is not necessary to take the $\|\cdot\|_s$. We have

$$\|f_1 * h_1\|_\infty \leqslant \|f_1\|_{r_1}\, \|h_1\|_{p_1}, \qquad p^{-1} + r_1^{-1} = 1$$

Now everything follows as before. We get

$$|\mathcal{G}'(0)| \leqslant \|G^3\|_1\, \|f_1\|_{r_1}\, \|G\|_s\, \|f_2\|_{r_2} \cdots \|f_{k-1}\|_{r_{k-1}}\, \|G\|_s$$
$$\text{with} \quad r_1^{-1} + r_2^{-1} + \cdots + r_{k-1}^{-1} = (k-1) - (k-2)\, s^{-1} \tag{3.26}$$

leading to a "softer" condition on the sum over α_i,

$$\sum_{i=1}^{k-1} \alpha_i > k - 1 + \tilde{\varepsilon} \tag{3.27}$$

It is easy to see that, for a graph with two points associated to two (or more) fields, e.g., Fig. 4, we have

$$r_1^{-1} + r_2^{-1} + \cdots + r_{k-1}^{-1} = (k-2) - (k-3)\, s^{-1} \Rightarrow \sum_{i=1}^{k-1} \alpha_i > k - 2 + \tilde{\varepsilon} \tag{3.28}$$

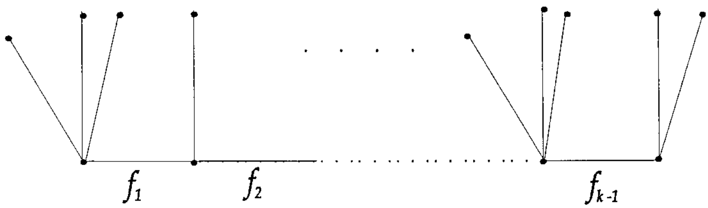

Fig. 4.

and so on. For $\tilde{p}$ points carrying two or more fields we have

$$\sum_{i=1}^{k-1} \alpha_i > k - \tilde{p} + \tilde{\varepsilon} \tag{3.29}$$

that is, it is easier (as expected) to control such graphs.

Now we study a graph with bifurcations, whose analysis will guide us to the general case. We will show that the bound above is maintained. We take the graph in Fig. 5, where (once more, dropping the prime out the notation)

$$\begin{aligned}
|\mathscr{G}'(0)| = \bigg| \int & G(x_1)\, f_1^{(x)}(x_1 - x_2)\, G(x_2)\, f_2^{(x)}(x_2 - x_2) \cdots f_{k_1}^{(x)}(x_{k_1} - x_0) \\
& \times G(y_1)\, f_1^{(y)}(y_1 - y_2) \cdots f_{k_2}^{(y)}(y_{k_2} - x_0) \\
& \times G(x_0)\, f_k^{(z)}(z_k - x_0)\, G(z_k)\, f_{k-1}^{(z)}(z_{k-1} - z_k) \\
& \times G(z_{k-1}) \cdots f_0^{(z)}(z_1 - z_0)\, G(z_0) \\
& \times G(u_1)\, f_1^{(u)}(u_1 - u_2)\, G(u_2) \cdots f_{k_3}^{(u)}(u_{k_3} - z_0) \\
& \times G(v_1)\, f_1^{(v)}(v_1 - v_2)\, G(v_2) \cdots f_{k_4}^{(v)}(v_{k_4} - z_0) \\
& \times dx_0\, dx_1 \cdots dx_{k_1}\, dy_1 \cdots dy_{k_2} \\
& \times dz_0\, dz_1 \cdots dz_k\, du_1 \cdots du_{k_3}\, dv_1 \cdots dv_{k_4} \bigg| \\
\equiv \int & \underbrace{[[[H^{(x)} H^{(y)} G] * f_k^{(z)}] \cdot G] * \cdots * f_0^{(z)}}_{\mathscr{H}} [G H^{(u)} H^{(v)}]
\end{aligned} \tag{3.30}$$

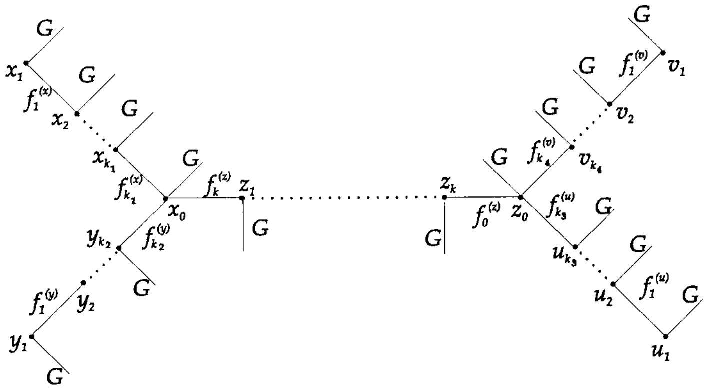

Fig. 5.

where $H^{(x)}(x_0) = \int G(x_1) f_1^{(x)} \cdots G(x_{k_1-1}) f_{k_1}^{(x)}(x_{k_1} - x_0)\, dx_1 \cdots dx_{k_1}$, G and f defined as before. Hence,

$$|\mathscr{G}'(0)| \leqslant \|\mathscr{H} G H^{(u)} H^{(v)}\|_1 \leqslant \|\mathscr{H}\|_q \|G\|_s \|H^{(u)}\|_{\gamma_3} \|H^{(v)}\|_{\gamma_4},$$
$$1 = q^{-1} + s^{-1} + \gamma_3^{-1} + \gamma_4^{-1}, \text{ i.e., } q^{-1} + \gamma_3^{-1} + \gamma_4^{-1} = 1 - s^{-1} \tag{3.31}$$

Repeating the previous analysis,

$$\|\mathscr{H}\|_q \leqslant \|(H^{(x)} H^{(y)} G) * f_k^{(z)}\|_{\tilde{q}} \|G\|_s \|f_{k-1}^{(z)}\|_{r_{k-1}} \cdots \|G\|_s \|f_0^{(z)}\|_{r_0},$$
$$r_0^{-1} + r_1^{-1} + r_2^{-1} + \cdots + r_{k-1}^{-1} + \tilde{q}^{-1} = q^{-1} + k - ks^{-1}$$

We also obtain

$$\|(H^{(x)} H^{(y)} G) * f_k^{(z)}\|_{\tilde{q}} \leqslant \|(H^{(x)} H^{(y)} G)\|_{\tilde{\gamma}} \|f_k^{(z)}\|_{r_k}, \qquad r_k^{-1} + \tilde{\gamma}^{-1} = \tilde{q}^{-1} + 1$$
$$\|(H^{(x)} H^{(y)} G)\|_{\tilde{\gamma}} \leqslant \|H^{(x)}\|_{\gamma_1} \|H^{(y)}\|_{\gamma_2} \|G\|_s, \qquad s^{-1} + \gamma_2^{-1} + \gamma_1^{-1} = \tilde{\gamma}^{-1}$$

Again, as analyzed before,

$$\|H^{(x)}\|_{\gamma_1} \leqslant \|G\|_s \|f_1^{(x)}\|_{r_1^x} \|G\|_s \|f_2^{(x)}\|_{r_2^x} \cdots \|G\|_s \|f_{k_1}^{(x)}\|_{r_{k_1}^x},$$
$$1/r_1^x + 1/r_2^x + \cdots + 1/r_{k_1}^x = 1/\gamma_1 + k_1 - k_1/s$$

(and similar expressions for $H^{(y)}$, $H^{(u)}$, $H^{(v)}$). Thus, it follows

$$|\mathscr{G}'(0)| \leqslant \prod_{\#\text{vertices}} (\|G\|_s) \prod_{\#\text{links}} (\|f\|_r) \tag{3.32}$$

(where # means number of) with

$$[1/r_0+1/r_1+\cdots+1/r_k]+[1/r_1^x+\cdots+1/r_{k_1}^x]+\cdots+[1/r_1^v+\cdots+1/r_{k_1}^v]$$
$$=2+k+k_1+\cdots+k_4-[2+k+k_1+\cdots+k_4]/s \tag{3.33}$$

That is, as for the first graph considered above,

$$\sum_i 1/r_i=(\#\text{vertices})-(\#\text{vertices})/s \tag{3.34}$$

where the sum is over all the $k-1$ links between neighbor vertices (the links are labeled by i in the expression above), leading to, as in (3.22),

$$\sum_{i=1}^{k-1} \alpha_i \geqslant k+\tilde{\varepsilon} \tag{3.35}$$

The analysis (and the formula) follows for any kind of tree with one field per vertice; with more fields we may still get improvements as already described.

Let us now make a precise statement of the results obtained above.

Theorem 3.3. The truncated correlation functions of the tridimensional Gross–Neveu model (with a smooth U.V. regularizer (2.1)) are given by (suppressing spinor indices)

$$S_{2m}(x_1,..., x_{2m})$$
$$=\delta_{1,m}\tilde{P}_n(x_1,x_2)-\int DY\prod_{i=1}^{2m}\{\tilde{G}_n(x_i,y_i)\}[\partial^{2m}_{y_1,..., y_{2m}}V_n](0)+R_{2m,n} \tag{3.36}$$

where

$$[\partial^{2m}_{y_1,..., y_{2m}}V_n](0)=\frac{\partial}{\partial\bar{\chi}(y_m)}\cdots\frac{\partial}{\partial\bar{\chi}(y_1)}V_n(\chi,\bar{\chi})\frac{\partial}{\partial\chi(y_{2m})}\cdots\frac{\partial}{\partial\chi(y_{m+1})}\bigg|_{\chi,\bar{\chi}\equiv 0}$$

V_n is the n step effective potential described by Theorem 3.1; $R_{2m,n}$ gives the contribution of the terms in V_n with more than $2m$ fields and goes to zero as $\Lambda, n\to\infty$, that is, just the kernels of V_n relating $2m$ fields survive in the $2m$-point function formula with the RG flow (and with the thermodynamic limit). DY above means $dy_1\cdots dy_p$, that is, in $y_1,..., y_{2m}$ there are only p distint points with p taking values between $m-1$ and $2m$ (details in

Theorem 3.1). Moreover, S_{2m} is analytic in λ, the parameter occuring in the initial interaction: $\lambda \int dx(\bar{\psi}_x \psi_x)^2$.

Specifically, for the two-point function

$$S_2(x_1, x_2) = \tilde{P}_n(x_1, x_2) + \tilde{S}_2(x_1, x_2) \tag{3.37}$$

with

$$\begin{aligned} \tilde{P}_\infty(x_1, x_2) &= b_\infty^{-1} D^{-1}(x_1, x_2) + \mathscr{C}_2(x_1, x_2) \\ |\mathscr{C}_2(x_1, x_2)| &\leqslant c/(1 + |x_1 - x_2|)^{d+1-\varepsilon} \end{aligned} \tag{3.38}$$

$\tilde{S}_2(x_1, x_2)$ as $n \to \infty$, is dominated by $c/(1 + |x_1 - x_2|)^{d-\varepsilon}$. In particular, $S_2 \to D^{-1}$ as $\lambda \to 0$ (and the difference $S_2 - D^{-1}$ is $\mathcal{O}(\lambda^2)$).

For the truncated four-point function, the term of lowest order in λ is given by

$$\lambda \int dy \prod_{i=1}^{4} \tilde{G}_n(x_i, y)$$

$$\text{where} \quad |\tilde{G}_\infty(x, y)| \leqslant b_\infty^{-1} D - 1(x, y) + c/(1 + |x - y|)^{d+1-\varepsilon} \tag{3.39}$$

For the general case $m \geqslant 2$, we have the bound below with long distance tree decay

$$|S_{2m}(x_1,..., x_{2m})| \leqslant c^m \sum_{\tau \in \{1, 2,..., 2m\}} B'_\tau \frac{1}{|\ell_1|^{\varepsilon_1}} \cdots \frac{1}{|\ell_{2m-1}|^{\varepsilon_{2m-1}}} \tag{3.40}$$

with B'_τ a combinatorial factor such that $\sum_\tau B'_\tau < c'^m$, where the sum $\sum_\tau$ is over all the tree graphs between $2m$ points $x_1, x_2,..., x_{2m}$, with branches $\ell_1,..., \ell_{2m-1}$ ($\ell_i = 1 + |x_{i_1} - x_{i_2}|$), and there is $\tilde{\varepsilon} > 0$ such that $0 < \varepsilon_i < \tilde{\varepsilon}$, for all i. And S_{2m} depends on λ as λ^{m-1} ($m \geqslant 2$; and so, rapidly vanishes as $\lambda \to 0$).

Now, we make some remarks.

The combinatorial factor B'_τ is directly related to B_τ from the effective potential (Theorem 3.1). In fact, the contribution of V_n to S_{2m} (3.36) considers all the kernels with $2m$ fields, i.e., it involves several kernels with different values of p (number of points), and different numbers of a (number of derivatives). From Theorem 3.1, Eq. (3.2), the contribution to the effective potential of the kernels with $2m$ fields, p points and a derivatives involves a sum over trees controlled by the factor B_τ (we have $\sum_\tau B_\tau < c^m$). We recall that for each tree in the effective potential we construct only one tree for the correlation function formula. Hence, to control $\sum_\tau B'_\tau$ we just have to know how many trees in the effective potential (thus, with their factors B_τ) appear with $2m$ fields. Recall also that, for each graph between p points in the formula (3.36) (connecting the effective potential and the correlation function) we have at most 3^p ways to distribute the ($2m$) fields among the p points (since we are studying the Gross–Neveu model with $N=1$). Thus, we have

$$\sum_\tau B'_\tau \leqslant \sum_{a=0}^{m} \sum_{p=\max\{m-1+2,2\}}^{2m} \sum_{r_1,\dots,r_{p-a}} \sum_{s_1,\dots,s_{p-a}} \sum_{\tau_p} B_{\tau_p}(m, a, \{r_j, s_j\})$$
$$\leqslant \sum_{a=0}^{m} \sum_{p=0}^{2m} c'^p \leqslant c^m$$

where r_j and s_j give us the number of fields in each point (see Theorem 3.1), and so, their sum is bounded by c^p.

A more precise analysis in the extraction of the factors ε_i shall permit us to get a condition better than $\varepsilon_i < \varepsilon$. Anyway, the bound (3.40) above is enough to show the long distance polynomial decay of the truncated correlation functions. And note that (3.38) proves that there is no mass generation in the tridimensional Gross–Neveu model (small λ, $N=1$).

A similar theorem is expected for the block RG description with minor changes (see remarks after Theorems 3.1 and 3.2, calculations and comments throughout this section).

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